Critical exponents in metastable decay via quantum activation

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We consider decay of metastable states of forced vibrations of a quantum oscillator close to bifurcation points, where dissipation becomes effectively strong. We show that decay occurs via quantum activation over an effective barrier. The decay probability W scales with the distance η to the bifurcation point as $|\ln W| \propto \eta^{\xi}$. The exponent ξ is found for a resonantly driven oscillator and an oscillator modulated at nearly twice its eigenfrequency.

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Decay of a metastable state is usually considered as resulting from tunneling or thermal activation. In this paper we study a different decay mechanism, quantum activation. It relates to systems far from thermal equilibrium. As tunneling, quantum activation is due to quantum fluctuations, but as thermal activation, it involves diffusion over an effective barrier separating the metastable state.

Metastable decay in nonequilibrium systems has attracted much attention recently in the context of switching between coexisting states of forced vibrations. Such diverse systems as trapped electrons and atoms [1, 2], Josephson junctions [3, 4], and nano- and micromechanical oscillators [5, 6] have been studied. The experiments largely focused on the parameter range where the system was close to a bifurcation point in which the metastable state disappears. In this range the decay probability is comparatively large and displays characteristic scaling with the distance to the bifurcation point. So far classical activation was studied, but recently quantum regime has been also reached [7].

For classical systems, scaling of the rate of activated decay near a bifurcation point was found theoretically both in the cases of equilibrium [8, 9, 10] and nonequilibrium systems [11, 12, 13]. In the latter case a scaling crossover may occur as the system goes from the underdamped to overdamped regime while approaching the bifurcation point [14]. Such crossover occurs also for quantum tunneling in equilibrium dissipative systems [15].

In this paper we study decay of metastable vibrational states in dissipative systems close to bifurcation points, where the motion becomes overdamped. The analysis refers to the systems of current interest, quantum oscillators driven by a resonant force or parametrically modulated at nearly twice the eigenfrequency. We show that at low temperatures decay occurs via quantum activation. The decay rate W scales with the distance to the bifurcation point η as $|\ln W| \propto \eta^{\xi}$. The scaling exponent is $\xi = 3/2$ for resonant driving, and $\xi = 2$ for parametric modulation; in addition, $|\ln W|$ displays a characteristic temperature dependence.

Quantum activation in periodically modulated systems can be understood by noting that metastable states are formed as a result of the balance between external driving and dissipation due to coupling to a thermal bath. For $T = 0$ dissipation corresponds to transitions to lower energy states with emission of excitations of the bath. However, modulated systems are more adequately described by the Floquet (quasienergy) states than by the energy eigenstates. Emission of bath excitations may result in transitions to both higher and lower quasienergies, albeit with different probabilities [16, 17]. The higher-probability transitions lead to relaxation towards a metastable state, whereas the lower-probability transitions lead to effective diffusion away from it, a finite-width distribution over quasienergy, and metastable decay. There is certain similarity here with the Unruh effect [18] where a uniformly accelerated relativistic detector coupled to a quantum zero-temperature field is described in its proper time by the Gibbs distribution with the acceleration-dependent temperature.

We will start with a resonantly driven nonlinear oscillator. Its Hamiltonian is

$$H_0(t) = \frac{1}{2}p^2 + \frac{1}{2}\omega_0^2 q^2 + \frac{1}{4}\gamma q^4 - qA\cos(\omega_F t). \qquad (1)$$

In the presence of weak damping the oscillator may have two coexisting stable states of classical forced vibrations [19]. They emerge already for a small modulation amplitude A provided the detuning $\delta\omega = \omega_F - \omega_0$ of the modulation frequency ω_F from the oscillator eigenfrequency ω_0 is small, $|\delta\omega| \ll \omega_F$. We assume that the nonlinearity is small, $|\gamma|\langle q^2\rangle \ll \omega_0^2$, and that $\gamma\,\delta\omega > 0$, which is necessary for the onset of bistability.

It is convenient to switch from q, p to slowly varying operators Q, P, using a transformation $q = C_{\rm res}(Q\cos\omega_F t + P\sin\omega_F t)$, $p = -C_{\rm res}\omega_F(Q\sin\omega_F t - P\cos\omega_F t)$ with $C_{\rm res} = (8\omega_F\delta\omega/3\gamma)^{1/2}$. The variables Q, P are the scaled coordinate and momentum in the rotating frame,

$$[P, Q] = -i\lambda, \qquad \lambda = 3\hbar\gamma/8\omega_F^2\,\delta\omega. \qquad (2)$$

The parameter λ plays the role of the effective Planck constant. We are interested in the semiclassical case; λ is the small parameter of the theory, $\lambda \ll 1$.

In the rotating wave approximation the Hamiltonian

(1) for $\delta\omega > 0$ becomes $H_0 = (\hbar/\lambda)\delta\omega\,\hat{g}$, with

$$\hat{g} \equiv g(Q,P) = \frac{1}{4}(Q^2+P^2-1)^2 - \beta^{1/2}Q, \qquad (3)$$
$$\beta = 3\gamma A^2/32\omega_F^3\,(\delta\omega)^3\,.$$

(for $\delta\omega < 0$ one should redefine $g \to -g, H_0 \to -(\hbar/\lambda)\delta\omega\, g$). The function g plays the role of the oscillator Hamiltonian in dimensionless time $\tau = t|\delta\omega|$. The eigenvalues of g give oscillator quasienergies.

The parameter β in Eq. (3) is the scaled intensity of the driving field. For weak damping the oscillator is bistable provided $0 < \beta < 4/27$. In this range the function $g(Q,P)$ has a shape of a tilted Mexican hat. The maximum at the top of the central dome and the minimum at the lowest point of the rim correspond, respectively, to the small- and large-amplitude states of forced vibrations. The saddle point of g corresponds to the unstable periodic state of the oscillator.

We will consider two major relaxation mechanisms of the oscillator: damping due to coupling to a thermal bath and dephasing due to oscillator frequency modulation by an external noise. Usually the most important damping mechanism is transitions between neighboring oscillator energy levels. They result from the coupling linear in the oscillator coordinate. Since the energy transfer is $\approx \hbar\omega_0$, in the rotating frame the transitions look instantaneous. We will assume that the correlation time of the noise that modulates the oscillator frequency is also short compared to $1/|\delta\omega|$, so that the noise is effectively δ-correlated in slow time τ. Then the quantum kinetic equation is Markovian in the rotating frame,

$$\dot{\rho} \equiv \partial_\tau \rho = i\lambda^{-1}[\rho, g] - \hat{\Gamma}\rho \; - \hat{\Gamma}^{\rm ph}\rho, \qquad (4)$$

where $\hat{\Gamma}\rho$ describes damping

$$\hat{\Gamma}\rho = \Gamma|\delta\omega|^{-1}\left[(\bar{n}+1)(\hat{a}^\dagger\hat{a}\rho - 2\hat{a}\rho\hat{a}^\dagger + \rho\hat{a}^\dagger\hat{a}) + \bar{n}(\hat{a}\hat{a}^\dagger\rho - 2\hat{a}^\dagger\rho\hat{a} + \rho\hat{a}\hat{a}^\dagger)\right], \qquad (5)$$

and $\hat{\Gamma}^{\rm ph}\rho$ describes dephasing,

$$\hat{\Gamma}^{\rm ph}\rho = \Gamma^{\rm ph}|\delta\omega|^{-1}\left[\hat{a}^\dagger\hat{a}, \left[\hat{a}^\dagger\hat{a}, \rho\right]\right]. \qquad (6)$$

Here, Γ and $\Gamma^{\rm ph}$ are the damping and dephasing rates, $\hat{a} = (2\lambda)^{-1/2}(Q+iP)$ is the lowering operator, and $\bar{n} = [\exp(\hbar\omega_0/kT) - 1]^{-1}$ is the oscillator Planck number. In what follows we use dimensionless parameters

$$\Omega = |\delta\omega|/\Gamma, \qquad \varkappa^{\rm ph} = \Gamma^{\rm ph}/\lambda\Gamma. \qquad (7)$$

We assume that $\varkappa^{\rm ph} \lesssim 1$. This means that the dephasing fluctuations intensity may be comparable to the intensity of quantum fluctuations associated with damping, which is $\propto \lambda\Gamma$, see below, but that $\Gamma^{\rm ph} \ll \Gamma$.

Metastable decay of the driven oscillator was studied earlier [16] assuming that the damping-induced broadening of quasienergy levels is small compared to the typical interlevel distance. This condition necessarily breaks near a bifurcation point where local extrema of $g(Q,P)$ come close to each other and the motion is slowed down. Therefore the analysis should be done differently. It is simplified in the Wigner representation of the density matrix,

$$\rho_W(Q,P) = \int d\xi e^{-i\xi P/\lambda}\rho\left(Q + \frac{1}{2}\xi, Q - \frac{1}{2}\xi\right), \qquad (8)$$

where $\rho(Q_1,Q_2) = \langle Q_1|\rho|Q_2\rangle$ is the density matrix in the coordinate representation. Using Eqs. (2)-(8) one can formally write the equation for ρ_W as a sum of terms proportional to different powers of λ,

$$\dot{\rho}_W = -\boldsymbol{\nabla}\left(\mathbf{K}\rho_W\right) + \lambda\hat{L}^{(1)}\rho_W + \lambda^2\hat{L}^{(2)}\rho_W. \qquad (9)$$

Here we introduced vectors $\mathbf{K} = (K_Q, K_P)$ and $\boldsymbol{\nabla} = (\partial_Q, \partial_P)$.

Vector $\mathbf{K}$ in Eq. (9) determines the evolution of the density matrix in the absence of quantum and classical fluctuations,

$$K_Q = \partial_P g - \Omega^{-1}Q \qquad K_P = -\partial_P g - \Omega^{-1}P. \qquad (10)$$

This evolution corresponds to classical motion

$$\dot{Q} = K_Q, \qquad \dot{P} = K_P. \qquad (11)$$

The condition $\mathbf{K} = \mathbf{0}$ gives the values of Q, P at the stationary states of the oscillator in the rotating frame.

The term $\hat{L}^{(1)}$ in Eq. (9) describes classical and quantum fluctuations due to damping and dephasing,

$$\hat{L}^{(1)} = \Omega^{-1}\left[\left(\bar{n}+\frac{1}{2}\right)\boldsymbol{\nabla}^2 + \varkappa^{\rm ph}\left(Q\partial_P - P\partial_Q\right)^2\right]. \qquad (12)$$

These fluctuations lead to diffusion in (Q,P)-space, as seen from the structure of $\hat{L}^{(1)}$.

The term $\hat{L}^{(2)}$ in Eq. (9) describes quantum effects of motion of the isolated oscillator,

$$\hat{L}^{(2)} = -\frac{1}{4}\left(Q\partial_P - P\partial_Q\right)\boldsymbol{\nabla}^2. \qquad (13)$$

In contrast to $\hat{L}^{(1)}$, the operator $\hat{L}^{(2)}$ contains third derivatives. Generally the term $\lambda^2\hat{L}^{(2)}\rho_W$ is not small, because ρ_W varies on distances $\sim\lambda$. However, it becomes small close to bifurcation points, as shown below.

From Eqs. (10), (11), for given damping Ω^{-1} the oscillator has two stable and one unstable stationary state in the rotating frame (periodic states of forced vibrations) in the range $\beta_B^{(1)}(\Omega) < \beta < \beta_B^{(2)}(\Omega)$ and one stable state outside this range [19], with

$$\beta_B^{(1,2)} = \frac{2}{27}\left[1 + 9\Omega^{-2} \mp \left(1 - 3\Omega^{-2}\right)^{3/2}\right]. \qquad (14)$$

At $\beta_B^{(1)}$ and $\beta_B^{(2)}$ the stable states with large and small Q^2+P^2, respectively (large and small vibration amplitudes), merge with the saddle state (saddle-node bifurcation). The values of Q, P at the bifurcation points 1,

2 are $Q_B = \beta_B^{-1/2} Y_B (Y_B - 1)$, $P_B = \beta_B^{-1/2} \Omega^{-1} Y_B$, where $Y_B = Q_B^2 + P_B^2$,

$$Y_B^{(1,2)} = \frac{1}{3}\left[2 \pm (1 - 3\Omega^{-2})^{1/2}\right]. \tag{15}$$

In the absence of fluctuations dynamics of a classical system near a saddle-node bifurcation point is controlled by one slow variable [20]. In our case it can be found by expanding $K_{Q,P}$ in $\delta Q = Q - Q_B$, $\delta P = P - P_B$, and the distance to the bifurcation point $\eta = \beta - \beta_B$. The function K_P does not contain linear terms in $\delta Q, \delta P$. Then, from Eq. (11), P slowly varies in time for small $\delta Q, \delta P, \eta$. On the other hand

$$K_Q \approx -2\Omega^{-1}\left(\delta Q - a_B \delta P\right), \quad a_B = \Omega(2Y_B - 1). \tag{16}$$

Therefore the relaxation time of Q is $\Omega/2$, it does not depend on the distance to the bifurcation point. As a consequence, Q follows P adiabatically, i.e., over time $\sim \Omega$ it adjusts to the instantaneous value of P.

The adiabatic approximation can be applied also to fluctuating systems. The approach is well known for classical systems described by the Fokker-Planck equation [21]. We now extend it to the quantum problem.

Formally we change in Eq. (9) from Q and P to $\delta\tilde{Q} = \delta Q - a_B \delta P$ and δP. For times $\tau \gg \Omega^{-1}$ the distribution ρ_W has a narrow peak as a function of $\delta\tilde{Q}$, whereas its dependence on δP is much more smooth. We seek ρ_W near its maximum over $\delta\tilde{Q}$ in the form

$$\rho_W = (2\pi\lambda\sigma^2)^{-1/2} \exp\left(-\delta\tilde{Q}^2/2\lambda\sigma^2\right) \bar{\rho}_W(\delta P), \tag{17}$$

where $\sigma^2 = \frac{1}{2}(1 + a_B^2)\left(\bar{n} + \frac{1}{2}\right) + \frac{1}{8}\varkappa^{\rm ph}\beta_B\Omega^2$. The $\delta\tilde{Q}$-dependent factor in ρ_W is chosen so that in Eq. (9) the term $\partial_{\delta\tilde{Q}} K_Q \rho_W$ and the term $\propto \lambda\partial^2_{\delta\tilde{Q}}\rho_W$ compensate each other. Note that corrections from $\lambda^2 \hat{L}^{(2)}\rho_W$ are of higher order in λ for $\delta\tilde{Q}^2 \lesssim \lambda$.

The function $\bar{\rho}_W$ describes the distribution over δP. In the spirit of the adiabatic approximation, it can be calculated disregarding small fluctuations of Q, i.e., setting $\delta\tilde{Q} = 0$ in Eq. (9). Formally, one obtains an equation for $\bar{\rho}_W$ by substituting Eq. (17) into the full kinetic equation (9) and integrating over $\delta\tilde{Q}$. This gives

$$\dot{\bar{\rho}}_W \approx \partial_P\left[\bar{\rho}_W \partial_P U + \lambda \mathcal{D}_B \partial_P \bar{\rho}_W\right], \tag{18}$$

where U and $\mathcal{D}$ have the form

$$U = \frac{1}{3}b(\delta P)^3 - \frac{1}{2}\beta_B^{-1/2}\eta\delta P, \qquad \eta = \beta - \beta_B,$$
$$\mathcal{D}_B = \Omega^{-1}\left[\left(\bar{n} + \frac{1}{2}\right) + \frac{1}{2}\varkappa^{\rm ph}(1 - Y_B)\right] \tag{19}$$

with $b = -\beta_B^{1/2}(2Y_B)^{-1}(1 - 2\Omega^2 Y_B + \Omega^2)$. In Eqs. (18), (19) we kept only the lowest order terms in $\delta P, \beta - \beta_B, \lambda$. In particular we dropped the term $-\lambda^2 Q_B \partial_P^3 \bar{\rho}_W/4$ which comes from the operator $\hat{L}^{(2)}$ in Eq. (9). One can show that, for typical $|\delta P| \sim |\eta|^{1/2}$, this term leads to corrections $\sim \eta, \lambda$ to $\bar{\rho}_W$.

Eq. (18) has a standard form of the equation for classical diffusion in a potential $U(\delta P)$, with diffusion coefficient $\lambda\mathcal{D}_B$. For $\eta b > 0$ the potential U has a minimum and a maximum. They correspond to the stable and saddle states of the oscillator. The distribution ρ_W has a diffusion-broadened peak at the stable state. Diffusion also leads to escape from the stable state, i.e., to metastable decay. The decay rate W is given by the Kramers theory [22],

$$W = Ce^{-R_A/\lambda}, \qquad R_A = \frac{2^{1/2}|\eta|^{3/2}}{3\mathcal{D}_B|b|^{1/2}\beta_B^{3/4}}, \tag{20}$$

with prefactor $C = \pi^{-1}(b\eta/2)^{1/2}\beta_B^{-1/4}|\delta\omega|$ (in unscaled time t).

The rate (20) displays activation dependence on the effective Planck constant λ. The characteristic quantum activation energy R_A scales with the distance to the bifurcation point $\eta = \beta - \beta_B$ as $\eta^{3/2}$. This scaling is independent of temperature. However, the factor $\mathcal{D}_B$ in R_A displays a characteristic T dependence. In the absence of dephasing we have $\mathcal{D}_B = 1/2\Omega$ for $\bar{n} \ll 1$, whereas $\mathcal{D}_B = kT/\hbar\omega_0\Omega$ for $\bar{n} \gg 1$. In the latter case the expression for W coincides with the result [11].

In the limit $\Omega \gg 1$ the activation energy (20) for the small-amplitude state has the same form as in the range of β still close but further away from the bifurcation point, where the distance between quasienergy levels largely exceeds their width [16]. We note that the rate of tunneling decay for this state is exponentially smaller; the tunneling exponent for constant quasienergy scales as $\eta^{5/4}$ [12], which is parametrically larger than $\eta^{3/2}$ for small η [for comparison, for a particle in a cubic potential (19) the tunneling exponent in the strong-damping limit scales as η [15]].

For the large-amplitude state the quantum activation energy, Eq. (20), displays different scaling from that further away from the bifurcation point, where $R_A \propto \beta^{1/2}$ for $\Omega \gg 1$ [16]. For this state we therefore expect a scaling crossover to occur with varying β.

The approach to decay of vibrational states can be extended to a parametrically modulated oscillator. The Hamiltonian of such an oscillator is

$$H_0(t) = \frac{1}{2}p^2 + \frac{1}{2}q^2\left[\omega_0^2 + F\cos(\omega_F t)\right] + \frac{1}{4}\gamma q^4. \tag{21}$$

When the modulation frequency ω_F is close to $2\omega_0$, as a result of parametric resonance the oscillator may have two stable states of vibrations at frequency $\omega_F/2$ (period-two states) shifted in phase by π [19]. For $F \ll \omega_0^2$ the oscillator dynamics is characterized by the dimensionless frequency detuning μ, effective Planck constant λ, and

relaxation time ζ,

$$\mu = \frac{\omega_F(\omega_F - 2\omega_0)}{F}, \quad \lambda = \frac{3|\gamma|\hbar}{F\omega_F}, \quad \zeta = \frac{F}{2\omega_F\Gamma}. \quad (22)$$

As before, λ will be the small parameter of the theory.

Parametric excitation requires that the modulation be sufficiently strong, $\zeta > 1$. For such ζ the bifurcation values of μ are

$$\mu_B^{(1,2)} = \mp(1 - \zeta^{-2})^{1/2}, \qquad \zeta > 1. \quad (23)$$

If $\gamma > 0$, as we assume, for $\mu < \mu_B^{(1)}$ the oscillator has one stable state; the vibration amplitude is zero. As μ increases and reaches $\mu_B^{(1)}$ this state becomes unstable and there emerge two stable period two states (a supercritical pitchfork bifurcation). They remain stable for larger μ. In addition, when μ reaches $\mu_B^{(2)}$ the zero-amplitude state also becomes stable (a subcritical pitchfork bifurcation). The case $\gamma < 0$ is described by replacing $\mu \to -\mu$.

The classical fluctuation-free dynamics for μ close to μ_B is controlled by one slow variable [20]. The analysis analogous to that for the resonant case shows that, in the Wigner representation, fluctuations are described by one-dimensional diffusion in a potential, which in the present case is quartic in the slow variable. The probability W of switching between the period-two states for small $\mu - \mu_B^{(1)}$ and the decay probability of the zero-amplitude state for small $\mu - \mu_B^{(2)}$ have the form $W = C\exp(-R_A/\lambda)$ with

$$R_A = |\mu_B|\eta^2/2(2\bar{n} + 1), \qquad \eta = \mu - \mu_B \quad (24)$$

($\mu_B = \mu_B^{(1,2)}$). The corresponding prefactors are $C_B^{(2)} = 2C_B^{(1)} = 2^{1/2}\pi^{-1}\Gamma\zeta^2|\mu_B||\mu - \mu_B|$. We note that dephasing does not affect the decay rate, to zeroth order in $\mu - \mu_B$.

From Eq. (24), at parametric resonance the quantum activation energy R_A scales with the distance to the bifurcation point as η^2. In the limit $\zeta \gg 1$ the same expression as Eq. (24) describes switching between period-two states still close but further away from the bifurcation point, where the distance between quasienergy levels largely exceeds their width. The exponent for tunneling decay in this case scales as η [17].

It follows from the above results that, both for resonant and parametric modulation, close to bifurcation points decay of metastable vibrational states occurs via quantum activation. It results from diffusion over a barrier. The quantum activation energy is smaller than the tunneling exponent. Near bifurcation points these quantities become parametrically different and scale as different powers of the distance to the bifurcation point.

The exponent of the decay rate displays a characteristic dependence on temperature. In the absence of dephasing, for $kT \gg \hbar\omega_0$ we have standard thermal activation, $R_A \propto 1/T$. The low-temperature limit is described by the same expression with kT replaced by $\hbar\omega_0/2$. Quantum activation imposes a limit on the sensitivity of bifurcation amplifiers based on modulated Josephson oscillators used for quantum measurements [3, 4].

In conclusion, we have studied decay of metastable states of forced vibrations of a quantum oscillator. Both energy dissipation from coupling to a bath and noise-induced dephasing were taken into account. We have found the exponent and the prefactor in the decay rate near bifurcation points. The quantum activation energy for resonantly excited period one states scales with the distance η to the bifurcation point as $\eta^{3/2}$, whereas for parametrically excited period two states it scales as η^2.

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[1] L. J. Lapidus, D. Enzer, and G. Gabrielse, Phys. Rev. Lett. **83**, 899 (1999).
[2] K. Kim et al., Phys. Rev. A **72**, 053402 (2005).
[3] I. Siddiqi et al., Phys. Rev. Lett. **94**, 027005 (2005).
[4] I. Siddiqi, R. Vijay, F. Pierre, C. M. Wilson, L. Frunzio, M. Metcalfe, C. Rigetti, and M. H. Devoret, cond-mat/0507248 (2005).
[5] J. S. Aldridge and A. N. Cleland, Phys. Rev. Lett. **94**, 156403 (2005).
[6] C. Stambaugh and H. B. Chan, cond-mat/0504791 (2005).
[7] M. Devoret, private communication.
[8] J. Kurkijärvi, Phys. Rev. B **6**, 832 (1972).
[9] R. Victora, Phys. Rev. Lett. **63**, 457 (1989).
[10] A. Garg, Phys. Rev. B **51**, 15592 (1995).
[11] M. I. Dykman and M. A. Krivoglaz, Physica A **104**, 480 (1980).
[12] A. P. Dmitriev and M. I. Dyakonov, Zh. Eksper. Teor. Fiz. **90**, 1430 (1986).
[13] O. A. Tretiakov and K. A. Matveev, Phys. Rev. B **71**, 165326 (2005).
[14] M. I. Dykman, I. B. Schwartz, and M. Shapiro, Phys. Rev. E **72**, 021102 (2005).
[15] A. O. Caldeira and A. J. Leggett, Ann. Phys. (N.Y.) **149**, 374 (1983).
[16] M. I. Dykman and V. N. Smelyansky, Zh. Eksper. Teor. Fiz. **94**, 61 (1988).
[17] M. Marthaler and M. I. Dykman, Phys. Rev. A **73**, 042108 (2006).
[18] W. G. Unruh, Phys. Rev. D **14**, 870 (1976).
[19] L. D. Landau and E. M. Lifshitz, *Mechanics* (Elsevier, Amsterdam, 2004), 3rd ed.
[20] J. Guckenheimer and P. Holmes, *Nonlinear Oscillators, Dynamical Systems and Bifurcations of Vector Fields* (Springer-Verlag, New York, 1987).
[21] H. Haken, *Synergetics: Introduction and Advanced Topics* (Springer-Verlag, Berlin, 2004).
[22] H. Kramers, Physica (Utrecht) **7**, 284 (1940).